Is the 3x+1 PROBLEM Intractal

by Kwek Keng Huat

"Mathematics is not yet ready for such problems . . ."

The 3*x* + 1 problem, also known as the $3x + 1$ conjecture, the Collatz problem, Syracuse problem, Kakutani's problem, Hasse's algorithm, Ulam's problem and hailstone number problem, concerns the behaviour of the iterates of the function f which takes odd integers n to $3n + 1$ and even integers n to $n/2$. Like many celebrated Diophantine equations, the $3x + 1$ Conjecture is so simple to state and yet intractably hard to solve.

The $3x + 1$ Conjecture asserts that, starting from any positive integer n, repeated iterations of the function eventually reaches the value 1. For example,

$f(5) = 16,\ f^{(2)}(5) = f(f(5)) = f(16) = 8,\ f^{(3)}(5) = f(f^{(2)}(5)) = 4,\ f^{(4)}(5) = f(f^{(3)}(5)) = 2,\ f^{(5)}(5) = f(f^{(4)}(5)) = 1.$

The $3x + 1$ problem was originally attributed to Lothar Collatz. Collatz circulated this problem at the International Congress of Mathematicians in 1950 in Cambridge, Massachusetts. Since then it has become the focus of rapidly increasing attention. Prizes have been offered for its solution: US$50 by H.S.M. Coxeter in 1970, then US$500 by Paul Erdös, and more recently £1000 by B. Thwaites; and a great number of false proofs have unsuccessfully chased the prize money.

The known results on the $3x + 1$ problem can be expressed in terms of the iterations of the function f, where

$$f(n) = \begin{cases} 3n + 1, & \text{if } n \text{ is an odd integer,} \\ n/2, & \text{if } n \text{ is an even integer.} \end{cases}$$

We call the sequence of iterates $n, f(n), f^{(2)}(n), \ldots,$ the trajectory of n. There are three possible outcomes for such trajectories when n is positive.

(1) Convergent trajectory. In this case some $f^{(k)}(n) = 1$.

(2) Non-trivial cycle trajectory. In this case the sequence $f^{(k)}(n)$ eventually becomes periodic and $f^{(k)}(n)$ 1 for any k 1.

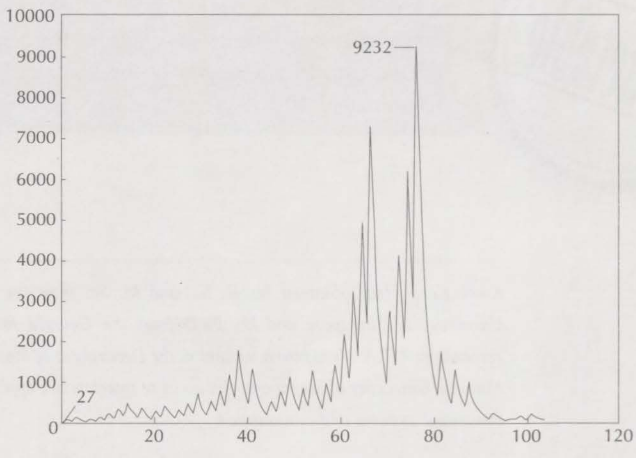
(3) Divergent trajectory. In this case $\lim_{k\to\infty} f^{(k)}(n) = \infty$.

The $3x + 1$ Conjecture asserts that when n is positive, the trajectory of n is convergent. Let us call the least positive integer k for which $f^{(k)}(n) = 1$ the halting time $T(n)$ of n; and set $T(n) = \infty$ if no such k exists. Then the $3x + 1$ Conjecture can be restated in terms of halting time as follows: Every positive integer n greater than 1 has a finite halting time $T(n)$.

Let us start with a number, say 7, and apply the function f to it until we reach 1:

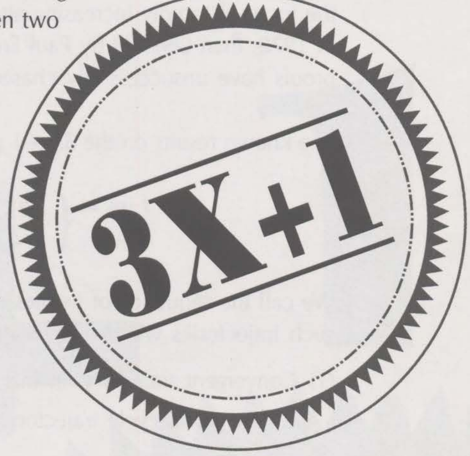
7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.

We see that $T(7) = 16$, and the sequence reaches a respectable high value of 52. Let us try another starting number 27. The following figure indicates the trajectory of 27.



We see that $T(27) = 111$, and the trajectory reaches a peak of 9232 at the 77th step. From the above two examples it shows that the computation of the $3x + 1$ sequence has probably consumed more CPU time than any other number theoretic conjecture. In fact, the $3x + 1$ Conjecture has been numerically checked for all n less than $2^{40} (\approx 1.2 \times 10^{12})$ by Nabua Yoneda at the University of Tokyo in 1983.

It seems that the route to prove the $3x + 1$ Conjecture is still far. However, the following heuristic probabilistic argument supports the Conjecture. Pick an odd integer n_0 at random and iterate the function f until another odd integer n_1 occurs. Then the probability that $n_1 = (3n_0 + 1)/2$ is 1/2, the probability that $n_1 = (3n_0 + 1)/4$ is 1/4 and so on. If one supposes that the function f is sufficiently "mixing" (a term used in Ergodic theory) [see *]that successive odd integers in the trajectory of n were drawn at random from the set of odd integers up to $2^k - 1$ for all k, then the expected growth in size between two consecutive odd integers in such a trajectory is



$$(3/2)^{1/2}\ (3/4)^{1/4}\ (3/8)^{1/8}\ (3/16)^{1/16} \ldots = \frac{3^{1/2+1/4+1/8+1/16+\ldots}}{2^{1/2+2/4+3/8+4/16+\ldots}}$$

$$= 3/4 < 1.$$

So this heuristic argument suggests that on the average the iterates in a trajectory tend to shrink in size, so that divergent trajectories should not occur. It also suggests an estimate of the halting time $T(n)$:

$$n(3/4)^{T(n)} \approx 1,$$
$$T(n) \approx \log n / \log(4/3).$$

The above assumption that $f(n)$ has some "mixing" properties is much weaker than what one needs to settle the $3x + 1$ Conjecture. The existing general methods in number theory and ergodic theory do not seem to be able to handle the problem. It seems that the $3x + 1$ problem is intractable at present. Paul Erdös regards that "Mathematics is not yet ready for such problems". Nevertheless, the study of the $3x + 1$ problem has uncovered a number of interesting phenomena; the further study of it may lead to discovery of other new phenomena. M^2

References

[1] M. Gardner, *Mathematical Games,* Scientific American 226 (1972), 114-118.

[2] J. C. Lagarias, *The 3x + 1 problem & its generalizations,* American Mathematical Monthly 92(1)(1985), 3-23.

[3] M. E. Lines, *Think of a number - Ideas, concepts and problems which challenge the mind and baffle the experts,* Adam Hilger 1990.

***Mixing**

Consider an incompressible fluid which consists of 30% ice-cream and 70% coca cola in a container. After sufficiently many repetitions of stirring, the percentage of ice-cream in any region of the container will be approximately 30%.

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